# Wave-front sensing by use of a Green's function solution to the intensity transport equation: reply to comment

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A recent comment has pointed out that some practically important aberration modes have zero curvature and in consequence seem difficult to sense using phase-diversity approaches that are equivalent to curvature sensing. Here we comment on the approaches that should be adopted when faced with a need to measure such aberration modes. © 2007 Optical Society of America

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## 1. INTRODUCTION

In a comment elsewhere in this issue [1] C. Campbell points out that certain practically important wavefront aberration modes, notably first-order astigmatism and trefoil (labels used for these modes vary, but adopting the descriptions of the polynomials specified, these match C. Campbell's descriptions using $Z_2^2$ and $Z_2^{-2}$, for example), have a zero Laplacian and thus no curvature. The author comments that the first-order astigmatic mode is an important aberration in ophthalmology but claims, incorrectly, that the lack of curvature means that these modes cannot therefore be sensed using the Green's theorem-based data analysis proposed by two of the authors of this reply [2]. The data characteristic through which aberration modes with zero Laplacian can be sensed was, in fact, noted in the comment [1] but was clearly not felt to provide a suitable sensing mechanism. We thank the author of the comment [1] for his observations, which appear to provide correct and useful analysis and to raise some interesting points. However, we disagree with the conclusion that he draws from his analysis—the ramifications of which are addressed in this brief reply—and wish to draw attention to some of the long-standing literature relevant to the phase-reconstruction challenge.

We divide this response into comments relating to systems in which the wavefront to be sensed is confined to a support with definite, hard edges and to those situations in which the support is soft edged.

## 2. HARD-EDGED APERTURES

The use of the wavefront sensing analysis system [2] based on Green's functions and applied to both first-order astigmatism and to trefoil in systems with a hard-edged aperture has, in fact, already been demonstrated experimentally by the authors in collaboration with colleagues [3]. Our comments here are therefore intended to highlight these results and to re-emphasize those properties of the data analysis [2] that render measurement of these aberration modes with zero curvature both feasible and straightforward.

The principle of phase-diverse wavefront sensing [4,5] and curvature sensing [6] as a special case are well established and form part of an extensive literature on wavefront reconstruction. In general, one has access only to an energy distribution (intensity function) in optics, and there are many occasions in which the interpretation of the optical field requires that the underlying complex amplitude function be known. For one-dimensional problems it is well understood that there are, in general, a countable infinity of solutions [7] and that either *a priori* information [8,9] about the optical field or a second intensity measurement are required in order to reconstruct the complex-valued optical field from the available energy measurements. Measure-theoretic arguments suggest that for two-dimensional problems the reconstruction of phase from a single energy distribution is almost always [10] unique (that is, ambiguities exist only in the case of functions forming a set of zero measure). However, simple counterexamples can easily be formulated [11], and in most practical cases reconstructions are made from measurements of two intensity distributions both of which related to the underlying complex amplitude in different and defined ways. Measurements reliant on two defocus values, such as wavefront curvature, are special cases of this general principle.

Two-defocus measurements can be explained from a geometric optics interpretation, in that the defocus is equivalent to measurement of the wave function on two planes along the propagation axis. In this description the concave (convex) regions of wavefront converge (diverge) as the wavefront propagates, leading to a positive (negative) axial intensity gradient measured in the direction of propagation. The wavefront-curvature technique exploits this by estimating the local wavefront curvature from the axial intensity gradient estimated from the difference between the defocused images. The data processing approach is encapsulated in Eq. (4) of [2] and is restated in [1].

In seeking a solution to a differential equation of this form one has to remember that this must be posed as a boundary-value problem and that elements of the general solution must depend on the boundary values. For this reason it has long been recognized that the practical implementation of wave-function reconstruction through the intensity transport equation [4,5] must provide information about the radial slope of the wavefront at the pupil boundary, and edge sensors to achieve this were suggested explicitly in [4,5] and implicitly in [6]. The solution utilized by Roddier [6] estimated the radial slope at the pupil edge through the use of a segmented ring of detectors. For those wavefront error modes with zero Laplacian the boundary conditions are the *only* component that defines the form of the solution, the contributions from within the pupil providing a null signal, as noted by C. Campbell [1]. Despite the fact that Campbell notes [1] that other signals may be generated "… at the perimeter of an aperture…" he continues "… perhaps, in some way not clear from their paper, Woods and Greenaway feel that the Green's function used codes this information into their expression … [but it] cannot add information not present in the measured data."

The solution proposed by Woods and Greenaway [2] does include an estimate of the radial slope of the wavefront at the pupil boundary, and discussion of this point was the purpose of Section 3 of the paper, especially the section from just before Eq. (10) to just after Eq. (13). Indeed, the principal thrust of the paper [2] is that this boundary-value information can be extracted from an area integral that covers the pupil boundary and does not require special-purpose detection schemes in order to achieve that. It is evident that this benefit of the proposed implementation was insufficiently clear and, for that reason, we will briefly reiterate the main points of that discussion.

As correctly noted [1] the formulation used [2] to develop the Green's function is dependent on the geometry of the problem and not on the data. In propagation between the two measurement planes from which the curvature data is derived, the radial wavefront slope at the aperture edge is manifest through a displacement of that edge radially in the direction of the optic axis (where the gradient on the first plane is directed inwards) or away from the optic axis (where the radial wavefront slope at the aperture edge in the first plane points away from the optical axis). Thus, and in the small-signal approximation, a discontinuity at the pupil edge leads to a signal the strength and arithmetic sign of which are dependent on the value and direction of the radial wavefront slope at the pupil edge. This signal appears as an encircling discontinuity, the locus of which follows the pupil edge. It is this that reconstructs, from the area integral, the line integral that should be present in order to represent the boundary conditions. The way in which those data are incorporated in the solution is encapsulated [2] in Eqs. (10)–(13) and demonstrated in the experimental tests previously published [3]. Figure 1(b) here shows a computer simulation of the signal produced at the aperture edge for astigmatism mode $Z_2^{-2}$ with a uniformly illuminated pupil [input wavefront phase shown in Fig. 1(a)].

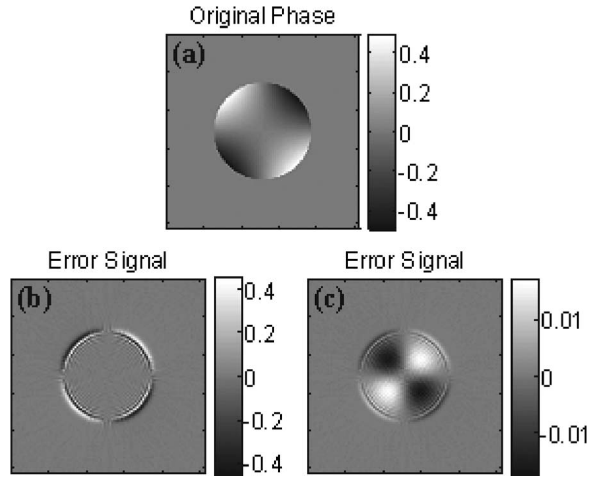

Fig. 1. (a) shows an astigmatic wavefront that is used to generate an error signal using phase diversity based on defocus. In (b) the error signal generated from a uniformly illuminated (i.e., hard-edged) pupil is shown. Note that the error signal is confined to the edge. For a soft-edged pupil (in this case a Gaussian profile with 10% drop in flux from center to rim) the error signal shown in (c) is uniformly distributed but of lower amplitude.

## 3. SOFT-EDGED APERTURES

Systems in which the intensity is not constant within the pupil still generate signals within a phase-diverse measurement system. The necessary and sufficient conditions for the generation of an error signal whenever the wavefront is not plane (i.e. constant phase) have been discussed in [12]. In general practical problems in metrology (including ophthalmology) rely on the measurement of coherent wavefronts emanating from laser sources. This leads to the generation of scintillation in the measurements, violating assumptions on which the Green's function approach is predicated, since the lateral gradient term is discarded in the analysis presented in Woods and Greenaway (and elsewhere, although [5] provides a solution for a Gaussian beam). However, analysis [[12]] shows that a wavefront that is not plane always produces a signal error, modulo $\pi$ radians (which implies ambiguity in the measurement of some discontinuous distributions) whether or not the pupil-plane intensity distribution is uniform within the pupil boundaries.

Zernike polynomials of the form discussed by C. Campbell [1] present a potentially greater problem when dealing with soft-edged distributions, such as a Gaussian-profile [5] laser beam. Here we note a problem with experimental data of deciding where to delineate the radius of the unit disc on which the Zernike polynomials are defined. In the algorithm described in [2] the discontinuity delineates the pupil edge and the pupil (unit disc) radius is thereby defined. However, in soft-edged problems or in problems where a noncircular pupil is involved, this approach is unsatisfactory. The comment made thus highlights a problem that exists in circumstances slightly different from those described by the author [1]. Although it should be noted that other phase-diversity methods satisfying the conditions described in [12] can provide signals not dependent on wavefront curvature, and can thus sense zero-curvature wavefront components, the very definition of the Zernike polynomials discussed is predicated on a function with circular support (unit disc) and uniform amplitude within that disc. For wave functions in soft-edged systems, therefore, the Zernike polynomial description needs to be replaced with a more general description such as the Kahunen–Loeve decomposition. Nonetheless, the error signal produced by a zero-curvature wavefront deformation in the case of nonuniform illumination is not confined to the aperture edge but exists everywhere that the illumination level is changing, as shown in Fig. 1(c) [input wavefront phase shown in figure 1(a)]. Note that the signal is spread across the support of the wavefront but is of low value.

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